METHODS OF LIE ALGEBRAS AND ORBITS IN THE STUDY OF DIFFERENTIAL SYSTEMS CONCERNED WITH THE PROBLEMS OF ENERGY SAFETY OF THE REPUBLIC OF MOLDOVA

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Abstract: *In the paper are presented the differential equations systems, describing the continuous effect of energy safety indicators changes on economy safety indicators changes. The first invariant GL(n,R)-integrals presenting explicit connections among mentioned indicators are found for some of these systems.*

Keywords – *differential system, integral, energy safety*

1. POLYNOMIAL DIFFERENTIAL SYSTEMS, SOLUTIONS AND INTEGRALS

Consider multi-dimensional polynomial differential system written in the tensor form [1]:

$$\frac{dx^j}{dt} = \sum_{k\in A} a^j_{j_1 j_2 \cdots j_k} x^{j_1} x^{j_2} \ldots x^{j_k} \equiv P^j(x,a), \qquad (1)$$

$$(j, j_1, j_2, \ldots, j_k = \overline{1,n})$$

where the coefficient tensor $a^j_{j_1 j_2 \cdots j_k}$ $(k \in A)$ – is symmetrical in lower indices where the complete convolution hold, A- is a finite set of non-negative different integers, $x = (x^1, x^2, \ldots, x^n)$ - is the vector of phase variables and a - is the totality of the coefficients of right-hand sides of system the (1). All the coefficients and variables vary in the field of real numbers R.

Definition 1. *Call a solution of the system (1) the system of continuous functions*

$$x^1 = \varphi^1(t),\ \ x^2 = \varphi^2(t),\ \ldots,\ \ x^n = \varphi^n(t) \qquad (2)$$

on independent variable t, defined together with their first derivates on some interval from R, such that after substitution of them instead $x^1, x^2, \ldots, x^n$ in (1) we obtain the identity on the whole interval of definition with regard to t.

Definition 2. *If the function* $F(x,a)$ $\left(\left(\frac{\partial F}{\partial x^1}\right)^2 + \left(\frac{\partial F}{\partial x^2}\right)^2 + \ldots + \left(\frac{\partial F}{\partial x^n}\right)^2 \neq 0\right)$, *defined and continuous together with its partial derivates in some area of phase space of variables $x^1, x^2, \ldots, x^n$, after the substitution in it of some solution (2) became a constant with regard to t, then they say that equality*

$$F(x,a) = C \qquad \textbf{(3)}$$

is the first integral of the system (1).

It is known [2] that holds

Theorem 1. *Any first integral (3) of the system (1) satisfies the condition*

$$\Lambda(F) \equiv \sum_{j=1}^{k} P^j(x,a) \frac{\partial F}{\partial x^j} = 0 \qquad \textbf{(4)}$$

and vice versa, any function $F(x,a)$, satisfying the condition (4), is the first integral of the system (1).

It is showen in [2] that any system (1) has $n-1$ functional-independent first integrals, which compose the general integral of the system.

If the equality (3) from definition 2 we have $C = 0$, than

$$F(x,a) = 0 \qquad \textbf{(5)}$$

is the particular integral of the system (1). Than the criteria of existence of particular integral (5) for the system (1) can be written with the aid of Λ from (4) as follows

$$\Lambda(F) = F \cdot V, \qquad \textbf{(6)}$$

where V is some analytical function.

2. LIE ALGEBRA OF OPERATORS ADMITTED BY THE SYSTEM (1), ITS INVARIANT INTEGRALS AND ORBITS

Definition 3. *Call the linear space L on the field R Lie algebra if for any two of its elements X, Y the operation of commutation is defined $[X,Y]$, which returns the element from L (commutator of elements*

X, Y) *and satisfies the following axioms:*

1) bilinearity: for any $X, Y, Z \in L$ *and* $\alpha, \beta \in R$

$[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z],$

$[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z];$

2) anti-symmetry: for any $X, Y \in L$

$[X, Y] = -[Y, X];$

3) identity of Jacobi: for any $X, Y, Z \in L$

$\big[[X, Y], Z\big] + \big[[Y, Z], X\big] + \big[[Z, X], Y\big] = 0.$

Call a dimension of Lie algebra the dimension of vector space L and in case of finite dimension r the algebra is denoted by L_r and is known as finite algebra.

Further consider finite Lie algebras L_r with basis elements $X_1, X_2, ..., X_r$, which can be written as differential operators of the first order on coordinates of vector x and coefficients of system (1) as follows

$$X_\theta = \xi^j_\theta(x)\frac{\partial}{\partial x^j} + D_\theta \qquad (j = \overline{1,n};\ \ \theta = \overline{1,r}), \qquad \textbf{(7)}$$

where

$$D_\theta = \sum_{k \in A}\left[\eta^j_{j_1 j_2 \cdots j_k}(a)\right]_\theta \frac{\partial}{\partial a^j_{j_1 j_2 \cdots j_k}}$$

$$(j, j_1, j_2, ..., j_k = \overline{1,n};\quad \theta = \overline{1,r}). \qquad \textbf{(8)}$$

Assume that operators (7) are admitted by system (1), i.e., according to [3], their coordinates satisfy the system of defining equations

$$\left(\xi^i_\theta\right)_{x^k} P^k = \xi^j_\theta P^i_{x^j} + D_\theta(P^i)$$

$$(i, j, k = \overline{1,n};\quad \theta = \overline{1,r}), \qquad \textbf{(9)}$$

where the notations $\left(\xi^i_\theta\right)_{x^k} = \dfrac{\partial \xi^i_\theta}{\partial x^k}$ and $P^i_{x^j} = \dfrac{\partial P^i}{\partial x^j}$ are made.

According to Lie theory it follows that in this case operators (7)-(8) generate $r-$parametrical group of transformations G_r, admitted by the system (1) and which can be written as follows

$$\overline{x}^j = f^j(x, \alpha), \qquad \overline{a}^j_{j_1 j_2 \cdots j_k} = b^j_{j_1 j_2 \cdots j_k}(a, \alpha)$$

$$(j, j_1, j_2, ..., j_k = \overline{1,n};\ \ k \in A), \qquad \textbf{(10)}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_r)$ is the vector on r parameters.

The invariants and comitants of system (1) with the group G_r for $n \geq 2$ are defined in [1] and [4].

Definition 4. *According to [6] we say that integral (first or particular) of the differential system (1) is invariant integral of this system with group* G_r *, if it can be given with the aid of invariants and comitants of differential system with group* G_r .

According to [5] call such integrals as invariant G_r - integrals.

Denote by $E^N(a)$ the space of coefficients of the system (1), where N is the dimension of this space. Let $a \in E^N(a)$ and any element $q \in G_r$ is given by equalities (10), where $\alpha -$ is vector with r parameters and $r < N$.

Definition 5. *Call a* $G_r -$ *orbit of point* a *for system (1) the set* $O(a) = \{a(q) \mid q \in G_r\}$.

It is known from [5-6] that

$$\dim_R O(a) = rank(M_1), \qquad \textbf{(11)}$$

where the matrix M_1 is build on coordinate vectors of operators (8) and takes the form

$$M_1 = \left(\left[\eta^j_{j_1 j_2 \cdots j_k}(a)\right]_\theta\right)$$

$$(j, j_1, j_2, ..., j_k = \overline{1,n};\ \ k \in A;\quad \theta = \overline{1,r}).$$

As the $rankM_1 = r, r-1, ..., 1, 0,$ than $\dim_R O(a) = r, r-1, ..., 1, 0.$

3. DIFFERENTIAL SYSTEMS, CONCERNED WITH PROBLEMS OF ENERGY SAFETY AND THEIR INVARIANT $GL(n,R)-$INTEGRALS

Consider $n-$dimensional affine differential system written in general form as follows

$$\frac{dx^j}{dt} = a^j + a^j_\alpha x^\alpha \qquad (j, \alpha = \overline{1,n}). \qquad \textbf{(12)}$$

It is shown in [7] that for $n \leq 5$ and $a^\alpha_\alpha = 0$ this system connects continuous effect of energy safety indicators on economy safety indicators.

3.1 Case of system (12) with $n = 2$

The system can be written as follows

$$\frac{dx}{dt} = a + cx + dy, \qquad \frac{dy}{dt} = b + ex + fy. \qquad \textbf{(13)}$$

It is shown in [6] the widest linear group admitted by the system (13) is centro-affine group $GL(2,R)$. According to [6], the polynomial basis of invariants and comitants of this system with regard to group $GL(2,R)$ consists of the next polynomials:

$i_1 = c + f$, $i_2 = c^2 + 2de + f^2$,

$i_3 = -ea^2 + (c - f)ab + db^2$, $k_1 = -bx + ay$,

$k_2 = -ex^2 + (c - f)xy + dy^2$,

$$k_3 = -(ea + fb)x + (ca + db)y. \qquad \textbf{(14)}$$

From [6] we obtain that holds

Theorem 2. *The system (13) is placed on* $GL(2,R)$*-orbits with the maximal dimensions iff* $i_3 \neq 0$ *, where* i_3

is centro-affine invariant (14).

Remark that the orbits with the maximal dimension reflect the case of most general place for system (13), as they described by the inequality ($i_3 \neq 0$).

In [6] is given

Theorem 3. *With the aid of* $i_1 - i_3, k_1 - k_3$ *from (14) is made the decomposition of the set of* $GL(2,R)$*-orbits of the maximal dimension (*$i_3 \neq 0$*) of the set of coefficients and variables for system (13) on five non-intersecting invariant sets:*

$$M_1 = \{i_2 - i_1^2 = 0,\ i_1 \neq 0\},\quad M_2 = \{i_1 = i_2 = 0\},$$

$$M_3 = \{i_2 - i_1^2 \neq 0,\quad 2i_2 - i_1 = 0\},$$

$$M_4 = \{i_2 - i_1^2 \neq 0,\ 2i_2 - i_1^2 < 0,\ 2i_1 i_3 - -(i_2 - i_1^2)k_3 \neq 0\},$$

$$M_5 = \{i_2 - i_1^2 \neq 0,\ 2i_2 - i_1^2 > 0,\ 2i_1 i_3 - -(i_2 - i_1^2)k_3 \neq 0\},$$

and on the each set the corresponding first invariant $GL(2,R)$ *- integrals is found for the mentioned system.*

Remark 1. *For* $i_1 \neq 0$ *the system (13) on* $GL(2,R)$ *-orbits with maximal dimension 4 (*$i_3 \neq 0$*) has particular invariant* $GL(2,R)$*-integral*

$$\Phi \equiv (i_2 - i_1^2)k_2 + 2(i_2 k_1 - i_1 k_3 - i_3) = 0,$$

where $i_1 - i_3,\quad k_1 - k_3$ *are taken from (14).*

3.2 Case of system (12) with $n = 3$

In [4] is given the functional basis of comitants of system (12) with $n = 3$ with regard to group $GL(3,R)$:

$$\delta_1 = a^\alpha u_\alpha,\quad \delta_2 = \alpha^\alpha_\beta a^\beta u_\alpha,\quad \delta_3 = a^\alpha_\gamma a^\beta_\alpha a^\gamma u_\beta,$$
$$\delta_4 = a^\alpha_\gamma a^\beta_p a^\gamma_q u_\alpha u_\beta u_r \varepsilon^{pqr},\quad \kappa_1 = x^\alpha u_\alpha,$$
$$\kappa_2 = a^\alpha_\beta x^\beta u_\alpha,\quad \kappa_3 = a^\alpha_\gamma a^\beta_\alpha x^\gamma u_\beta,\quad \theta_1 = a^\alpha_\alpha,$$
$$\theta_2 = a^\alpha_\beta a^\beta_\alpha,\quad \theta_3 = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta, \tag{15}$$

where the coordinates of vector $u = (u_1, u_2, u_3)$ vary by the low of covariant vectors [8], and ε^{pqr} is the unit three-vector with coordinates $\varepsilon^{123} = -\varepsilon^{132} = \varepsilon^{312} = -\varepsilon^{321} = \varepsilon^{231} = -\varepsilon^{213} = 1$ and $\varepsilon^{pqr} = 0 \quad (p,q,r = \overline{1,3})$ in other cases.

It is shown that for equality $\dim_R O(a) = 9$ holds (maximal dimension of $GL(3,R)$-orbits for system (12) with $n = 3$) it is necessary that condition $\delta_4 \neq 0$ holds.

In [3] is proved

Theorem 4. *For* $\delta_4 \neq 0$ *and* $n = 3$ *the next canonical forms and corresponding first* $GL(3,R)$ *- integrals are found for the system (12):*

Case I.

Invariant condition:

$$\delta_3 \neq 0,\quad \theta_1 = \theta_2 = \theta_3 = 0.$$

Canonical form:

$$\frac{dx^1}{dt} = a + x^2,\quad \frac{dx^2}{dt} = b + x^3,\quad \frac{dx^3}{dt} = c.$$

First invariant $GL(3,R)$ *-integral:*

$$F_1 = 2(\delta_3\kappa_2 - \delta_2\kappa_3) - \kappa_3^2 = C_1.$$

Case II.

Invariant condition:

$$\delta_3 + \theta_1\kappa_3 \neq 0,\quad \theta_1 \neq 0,\quad \theta_2 = \theta_1^2,\quad \theta_3 = \theta_1^3.$$

Canonical form:

$$\frac{dx^1}{dt} = a + x^2,\quad \frac{dx^2}{dt} = b + x^3,\quad \frac{dx^3}{dt} = c + nx^3.$$

First invariant $GL(3,R)$ *-integral:*

$$F_1 = \theta_1^2\kappa_2 - \theta_1\kappa_3 + (\delta_3 - \delta_2\theta_1)\ln|\delta_3 + +\theta_1\kappa_3| = C_1.$$

Case III.

Invariant condition:

$$\theta_1 = \theta_3 = 0,\quad \theta_2 \neq 0.$$

Canonical form:

$$\frac{dx^1}{dt} = a + x^2,\quad \frac{dx^2}{dt} = b + x^3,\quad \frac{dx^3}{dt} = c + mx^2.$$

First invariant $GL(3,R)$ *-integral:*

$$F_1 = 2\theta_2(\delta_2 + \kappa_3)^2 - (2\delta_3 + \theta_2\kappa_2)^2 = C_1.$$

Case IV.

Invariant condition:

$$\theta_1 = \theta_2 = 0,\quad \theta_3 \neq 0.$$

Canonical form:

$$\frac{dx^1}{dt} = a + x^2,\quad \frac{dx^2}{dt} = b + x^3,\quad \frac{dx^3}{dt} = c + lx^1.$$

First invariant $GL(3,R)$ *-integral:*

$$F_1 = (3\delta_3 + \theta_3\kappa_1)^3 + 3\theta_3^2(\delta_1 + \kappa_2)^3 + +9\theta_3(\delta_2 + \kappa_3)^3 - 9\theta_3(3\delta_3 + \theta_3\kappa_1)(\delta_1 + \kappa_2)\cdot \cdot(\delta_2 + \kappa_3) = C_1.$$

3.2 Case of system (12) with $n = 4$

In this case the functional basis of comitants of system (12) with $n = 4$ with regard to the group $GL(4, R)$ can be written as follows

$\alpha_1 = a^{\alpha} u_{\alpha}$, $\alpha_2 = a^{\alpha}_{\beta} a^{\beta} u_{\alpha}$, $\alpha_3 = a^{\alpha}_{\gamma} a^{\beta}_{\alpha} a^{\gamma} u_{\beta}$,

$\alpha_4 = a^{\alpha}_{\delta} a^{\beta}_{\alpha} a^{\gamma}_{\beta} a^{\delta} u_{\gamma}$,

$\alpha_5 = a^{\alpha}_{p} a^{\beta}_{q} a^{\gamma}_{\beta} a^{\delta}_{r} a^{\mu}_{\delta} a^{\nu}_{\mu} u_s u_{\alpha} u_{\gamma} u_{\nu} \varepsilon^{pqrs}$,

$\beta_1 = x^{\alpha} u_{\alpha}$, $\beta_2 = a^{\alpha}_{\beta} x^{\beta} u_{\alpha}$,

$\beta_3 = a^{\alpha}_{\gamma} a^{\beta}_{\alpha} x^{\gamma} u_{\beta}$, $\beta_4 = a^{\alpha}_{\delta} a^{\beta}_{\alpha} a^{\gamma}_{\beta} x^{\delta} u_{\gamma}$, $\gamma_1 = a^{\alpha}_{\alpha}$,

$\gamma_2 = a^{\alpha}_{\beta} a^{\beta}_{\alpha}$, $\gamma_3 = a^{\alpha}_{\gamma} a^{\beta}_{\alpha} a^{\gamma}_{\beta}$, $\gamma_4 = a^{\alpha}_{\delta} a^{\beta}_{\alpha} a^{\gamma}_{\beta} a^{\delta}_{\gamma}$,

where the coordinates of vector $u = (u_1, u_2, u_3, u_4)$ vary by the low of covariant vectors [8], and ε^{pqrs} is the unit four-vector equal to 1 with even permutation of different upper indices, and equal to -1 with odd permutation of these indices, and $\varepsilon^{pqrs} = 0 \quad (p, q, r, s = \overline{1,4})$ in other cases.

It is shown that for equality $\dim_R O(a) = 16$ holds (maximal dimension of $GL(4, R)$-orbit for system (12) with $n = 4$) it is necessary that condition $\alpha_5 \neq 0$ holds.

It is proved

Theorem 5. *For* $\alpha_5 \neq 0$ *and* $n = 4$ *in some cases for the system (12) the following canonical forms are found and are obtained first* $GL(4, R)$*- integrals for the system (12):*

Case I.

Invariant condition:

$$\alpha_4 \neq 0, \quad \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0.$$

Canonical form:

$$\frac{dx^1}{dt} = A + x^2, \quad \frac{dx^2}{dt} = B + x^3, \quad \frac{dx^3}{dt} = C + x^4,$$

$$\frac{dx^4}{dt} = D.$$

First invariant $GL(4, R)$ *-integrals:*

$$F_1 = \beta_4^2 + 2\alpha_3\beta_4 - 2\alpha_4\beta_3 = C_1,$$

$$F_2 = 3\alpha_4\alpha_2\beta_4 - 3\beta_4\alpha_3^2 - 3\beta_4^2\alpha_3 - \beta_4^3 -$$
$$- 3\alpha_4^2\beta_2 + 3\alpha_3\alpha_4\beta_3 + 3\alpha_4\beta_4\beta_3 = C_2.$$

Case II.

Invariant condition:

$$\alpha_4 + \gamma_1\beta_4 \neq 0,$$

$$\gamma_1 \neq 0, \quad \gamma_2 = \gamma_1^2, \quad \gamma_3 = \gamma_1^3, \quad \gamma_4 = \gamma_1^4.$$

Canonical form:

$$\frac{dx^1}{dt} = A + x^2, \quad \frac{dx^2}{dt} = B + x^3, \quad \frac{dx^3}{dt} = C + x^4,$$

$$\frac{dx^4}{dt} = D + L_4 x^4.$$

First invariant $GL(4, R)$ *-integrals:*

$$F_1 = -\gamma_1\beta_4 + \gamma_1^2\beta_3 + (\alpha_4 - \alpha_3\gamma_1)\ln|\alpha_4 +$$
$$+ \gamma_1\beta_4 = C_1,$$

$$F_2 = \gamma_1^4\beta_3^2 + \gamma_1^2\beta_4^2 - 2\gamma_1\beta_4\alpha_4 + 4\gamma_1^2\beta_4\alpha_3 -$$
$$- 2\gamma_1^3\beta_4\alpha_2 + 2\gamma_1^3\beta_2\alpha_4 - 2\gamma_1^3\beta_4\beta_3 - 2\gamma_1^3\beta_3\alpha_3 +$$
$$+ 2\gamma_1^4\beta_3\alpha_2 - 2\gamma_1^4\beta_2\alpha_3 + 2(\alpha_4 - \alpha_3\gamma_1)\ln|\alpha_4 +$$
$$+ \gamma_1\beta_4| = C_2.$$

Case III.

Invariant condition:

$$\gamma_1 = \gamma_3 = 0, \quad \gamma_2 \neq 0, \quad \gamma_4 = \frac{1}{2}\gamma_2^2.$$

Canonical form:

$$\frac{dx^1}{dt} = A + x^2, \quad \frac{dx^2}{dt} = B + x^3, \quad \frac{dx^3}{dt} = C + x^4,$$

$$\frac{dx^4}{dt} = D - L_3 x^3.$$

First invariant $GL(4, R)$ *-integral:*

$$F_1 = -\beta_3^2\gamma_2 + 2\beta_4^2 - 4\beta_3\alpha_4 + 4\beta_4\alpha_3 = C_1.$$

REFERENCES

[1] Sibirsky K.S., *Introduction to the Algebraic Theory of Invariants of Differential Equations*, Kishinev, Shtiintsa, 1982 (in Russian, published in English in 1988)

[2] Pontryaghin L.S., *Ordinary differential equations*, Moscow, Nauka, 1974 (in Russian)

[3] Gherstega N., Popa M., *Lie algebras of the operators and three-dimensional polynomial differential systems*, Buletinul Academiei de Stiinte a Moldovei, Matematica, 2005 (to appear)

[4] Gherstega N., Popa M., *Mixed comitants and* $GL(3, R)$*-orbit's dimensions for the three-dimensional differential systems*, Buletinul Stiintific, Universitatea din Pitesti, Seria Matematica si Informatica, Nr.9 (2003), pp.149-154

[5] Ovsyannikov L.V., *Group analysis of differential equations*, Moscow, Nauka, 1978, English transl. by Academic Press, 1982

[6] Popa M., *Algebraic methods for differential systems*, Seria Matematica Aplicata si Industriala, Nr.15, Editura the Flower Power, Universitatea din Pitesti, 2004 (in Romanian)

[7] Bicova E., Methods of calculation and analysis of the indicators of energy safety. Academy of Sciences of Moldova, Series “Energy Safety of the Republic of Moldova” (book 2), Chishinau, 2005 (in Russian)

[8] Gurevich G.B., Foundations of the theory of algebraic invariants, GITTL, Moscow, 1948, English transl. Noordhoff, 1964